Linear metric perturbations in near-horizon extremal Kerr

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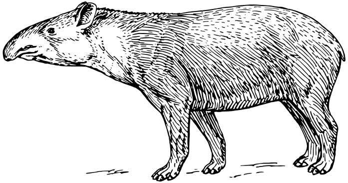

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Linearized gravity

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + \mathcal{O}(h^2)$$

Applications

- Dynamics of spacetime within or beyond GR
  e.g. stability of black hole
- Weak-field experimental tests of GR
  e.g. gravitational radiation

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The common tool is linearized Einstein field equation (LEE)

$$G^{(1)}_{\mu\nu}[h^{(1)}] = 8\pi T_{\mu\nu}$$

Hard to solve $\Longrightarrow$ need separation of variables

Metric perturbations in Kerr

Teukolsky's method

- Newman-Penrose formalism $\Longrightarrow$ perturbation equations for Ψ_4 or Ψ_0
- One master to rule them all (scalar, vector, tensor, spinor)
- Separation of variables ✓

Metric perturbations in Kerr

from curvature to metric

- Metric reconstruction by Chandrasekhar
    - *"The procedure is so complicated that it does not seem to have been used, at least in its entirety, in any application." — Saul Teukolsky*
- Hertz potentials
    - must be performed case by case
    - restricted to radiation gauge $\Longrightarrow$ singularities in presence of source

Complicated!

Go back to Schwarzschild

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Symmetry

- Static $\Longrightarrow$ time translation symmetry
- Spherical $\Longrightarrow$ rotation symmetry

Go back to Schwarzschild

$$\mathcal{M}^4 = \mathcal{M}^2 \times S^2$$

"$2+2$"-decomposition of the spacetime

- Scalar basis of $SO(3)$ acting on S^2 are just $Y_{lm}(\theta, \phi)$
- Vector basis $D_a Y_{lm}$, $\varepsilon_a{}^b D_b Y_{lm}$
- Tensor basis $D_{\langle a} D_{b\rangle} Y_{lm}$, $\Omega_{ab} Y_{lm}$, $\varepsilon_{\langle a}{}^c D_{b\rangle} D_c Y_{lm}$
- Separation of variables ✓

Question: Extended to Kerr?

Go back to Schwarzschild

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Question: Extended to Kerr?

Answer: Yes, but not for any Kerr.

Near-horizon extremal Kerr (NHEK)

The NHEK limit

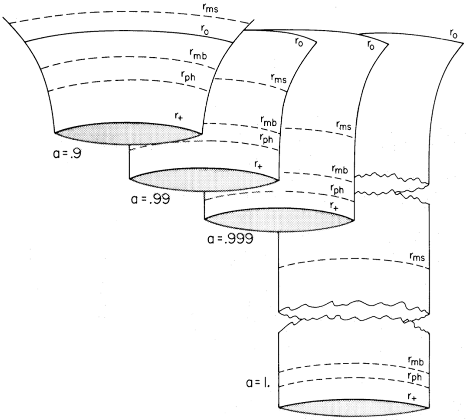
- Extremal condition

$$a = M$$

- Near-horizon limit
  - rescale the BL coordinates t, r, ϕ

$$\frac{t}{M} = \frac{2T}{\lambda}, \quad \frac{r}{M} = 1 + \lambda R, \quad \phi = \Phi + \frac{t}{2M}.$$

  - take the limit $\lambda \to 0$



What's new?

- Two additional Killing vectors

Kerr	NHEK	
∂_ϕ	∂_Φ	$= Q$
∂_t	∂_T	$= H_+$
	$T\,\partial_T - R\,\partial_R$	$= H_0$
	$(T^2 + \frac{1}{R^2})\,\partial_T - 2\,T R\,\partial_R - \frac{2}{R}\,\partial_\Phi$	$= H_-$

- Extended isometry group

$$SL(2,\mathbb{R}) \times U(1)$$

Isometry group

- $\{Q = \partial_\Phi\}$ generates $U(1)$
- $\{H_0, H_+, H_-\}$ generates $SL(2,\mathbb{R})$
- The Casimir element

$$\Omega = H_0(H_0 - 1) - H_- H_+$$

- Generators independent of $u(= \cos\theta) \implies$ Isometry acts on u-slices

From isometry group to basis decomposition

Schwarzschild

- "2+2"-decomposition
- $SO(3)$ acting on S^2
- 1 scalar, 2 vector, 3 tensor basis
- Y_{lm}, $D_a Y_{lm}$, etc.

NHEK

- "3+1"-decomposition
- $SL(2,\mathbb{R}) \times U(1)$ acting on M^3
- 1 scalar, 3 vector, 6 tensor basis
- ?

The highest-weight method

The method

- Simultaneous diagonalization of $\{Q, \Omega, \mathcal{L}_{H_0}\}$
- Label the states by m, h, k
- The highest weight $(k = 0)$ annihilated by $\mathcal{L}_{H_+}$
- All other basis obtained by applying $\mathcal{L}_{H_-}$ for k times

Scalar basis

- $k = 0$ (Highest)

$$F \propto R^h e^{im\Phi}$$

- $k = 1$

$$F \propto -2R^{h-1} e^{im\Phi} (hRT + im)$$

- $k = 2$

$$F \propto -2R^{h-2} e^{im\Phi} \Big(-2i(2h-1)mRT + h(1-2h)R^2T^2 + h + 2m^2 \Big)$$

Separation of variables

Scalar Laplacian

$$\Box\psi = \frac{1}{2M^2\Gamma}\left(\Omega\,\psi + \frac{(u^4+6u^2-3)}{4(1-u^2)}\mathcal{L}_Q^2\psi + \mathcal{L}_{\partial_u}\left[(1-u^2)\mathcal{L}_{\partial_u}\psi\right]\right)$$

- Separation of variables ✓
- T, R, Φ-pieces determined by symmetry, only u-dependence unknown
- u-dependence for free wave are spheroidal harmonics

Linearized Einstein equation

$$G^{(1)}h_{ab} = \overleftrightarrow{M}(T,R)\,\overleftrightarrow{D}_u^2[u]e^{im\Phi}$$

- Separation of variables ✓
- T, R, Φ-pieces determined by symmetry
- $\overleftrightarrow{D}_u^2[u] \equiv (\overleftrightarrow{A}\partial_u^2 + \overleftrightarrow{B}\partial_u + \overleftrightarrow{C})\vec{V}(u)$
- LEE $\Longrightarrow$ 10 ODEs for the polar angle $(u = \cos\theta)$!

Conclusions

- Highest weight method $\Longrightarrow$ separation of variables ✓
- Metric perturbation instead of curvature perturbation ✓
- Analytical solutions to LEE with source (e.g. dCS, EDGB) ✓
- Could be extended to near-extremal Kerr (future work)

Conclusions

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The End. Thank you.

*Additional slides

The NHEK limit

Kerr metric In Boyer-Lindquist (BL) coordinates t, r, θ, ϕ

$$ds^2 = -\frac{\Delta}{\Sigma}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\Sigma}{\Delta}\, dr^2 + \Sigma\, d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2 + a^2)\, d\phi - a\, dt\right]^2$$

The NHEK metric in the Poincaré coordinates $T, \Phi, R, u (= \cos\theta)$

$$ds^2 = 2M^2\Gamma\left[-R^2\, dT^2 + \frac{1}{R^2}\, dR^2 + \frac{1}{1-u^2}\, du^2 + \Lambda^2(d\Phi + R\, dT)^2\right]$$

$$\Gamma(u) = (1+u^2)/2 \quad \text{and} \quad \Lambda(u) = 2\sqrt{1-u^2}/(1+u^2)$$

Vector basis

- Casimir element in the space of vector functions

$$\Omega = \mathcal{L}_{H_0}(\mathcal{L}_{H_0} - 1) - \mathcal{L}_{H_-}\mathcal{L}_{H_+}$$

- Operators $\Longrightarrow$ Lie derivatives along K.Vs
- Three independent vector basis for each weight
- The general vector basis

$$V^{(m\,h\,k)} = \mathcal{L}_{H_-}^{k}\left(\left(C_T R^{h-1}\hat{e}_T + C_\Phi R^{h}\hat{e}_\Phi + C_R R^{h+1}\hat{e}_R\right) e^{im\Phi} \right)$$

Symmetric tensor basis

- Solve the same set of eqs as for vector cases
- Six independent tensor basis for each weight
- The general tensor basis

$$T_{AB}^{(m\,h\,k)} = \mathcal{L}_{H_-}^{k} \begin{pmatrix} C_{TT} R^{h+2} & C_{T\Phi} R^{h+1} & C_{TR} R^{h} \\ * & C_{\Phi\Phi} R^{h} & C_{\Phi R} R^{h-1} \\ * & * & C_{RR} R^{h-2} \end{pmatrix} e^{im\Phi}$$

Maxwell system

- Choose the three vector basis

$$V_{1a}^{(m,h,0)} = R^{h+1} dT$$
$$V_{2a}^{(m,h,0)} = R^{h} d\Phi$$
$$V_{3a}^{(m,h,0)} = R^{h-1} dR$$

- LHS of Maxwell equation for the highest weight

$$(\nabla_a \mathcal{F}^{ab})_{k=0} = \frac{1}{M^4}\left(G_0(u) n_a F^{(m,h,0)} + G_1(u) V_{1a}^{(m,h,0)} + G_2(u) V_{2a}^{(m,h,0)} + G_3(u) V_{3a}^{(m,h,0)} \right)$$

- Assume completeness, variable separate for each weight

Linearized Einstein equation

- Choose the six tensor basis

$$
\begin{aligned}
{W_1}_{ab}^{(m,h,0)} &= R^{h+2} dT \otimes dT \\
{W_2}_{ab}^{(m,h,0)} &= R^{h+1} dT \otimes d\Phi \\
{W_3}_{ab}^{(m,h,0)} &= R^{h} dT \otimes dR \\
{W_4}_{ab}^{(m,h,0)} &= R^{h} d\Phi \otimes d\Phi \\
{W_5}_{ab}^{(m,h,0)} &= R^{h-1} d\Phi \otimes dR \\
{W_6}_{ab}^{(m,h,0)} &= R^{h-2} dR \otimes dR
\end{aligned}
$$

Linearized Einstein equation

- Assume completeness, decompose the metric perturbation

$$h_{ab} = \sum_{m,\,h,\,k} h_{ab}^{(m\,h\,k)} = \sum_{m,\,h,\,k} \left(n_a n_b F^{(m\,h\,k)} f(u) + \sum_{i=1}^{3} n_{\{a} V_{i\,b\}}^{(m\,h\,k)} v_i(u) + \sum_{j=1}^{6} W_{j\,ab}^{(m\,h\,k)} t_j(u) \right)$$